

# 3d superconformal indices and isomorphisms of M2-brane theories

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**ABSTRACT:** We test several expected isomorphisms between the  $U(N) \times U(N)$  ABJM theory and  $(SU(N) \times SU(N))/\mathbb{Z}_N$  theory including the BLG theory by comparing their superconformal indices. From moduli space analysis, it is expected that this equivalence can hold if and only if the rank  $N$  and Chern-Simons level  $k$  are coprime. We also calculate the index of the ABJ theory and investigate whether some theories with identical moduli spaces are isomorphic or not .

**KEYWORDS:** Supersymmetric gauge theory, Chern-Simons Theories, M-theory.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Dual photon and moduli space of vacua</b>	<b>3</b>
<b>3. Magnetic charge and charge quantization condition</b>	<b>5</b>
<b>4. The superconformal index</b>	<b>6</b>
4.1 $U(N_1)_k \times U(N_2)_{-k}$ ABJ theory	7
4.2 $SU(N)_k \times SU(N)_{-k}$ theory	10
<b>5. Test of the conjectured isomorphisms</b>	<b>10</b>
5.1 $N = 2$	11
5.2 $N = 3$	13
<b>6. Search for a possibility of extended isomorphism with higher <math>k</math></b>	<b>14</b>
6.1 $U(2+l)_k \times U(2)_{-k}$ ABJ theory v.s. $(SU(2)_{k^2} \times SU(2)_{-k^2})/\mathbb{Z}_2$ BLG theory	14
6.2 $U(2+l)_k \times U(2)_{-k}$ ( $0 \leq l \leq  k $ ) ABJ theory v.s. $SU(2)_{k^2/2} \times SU(2)_{-k^2/2}$ BLG theory	15
<b>7. Conclusions</b>	<b>16</b>
<b>A. partition function of <math>SU(2)_k \times SU(2)_{-k}</math> ABJM theory</b>	<b>17</b>
<b>B. Full result</b>	<b>18</b>
B.1 $U(2) \times U(2)$ ABJM theory	18
B.2 $(SU(2) \times SU(2))/\mathbb{Z}_2$ BLG theory	19
B.3 $U(3) \times U(3)$ ABJM theory	20
B.4 $(SU(3) \times SU(3))/\mathbb{Z}_3$ theory	22
B.5 $U(3) \times U(2)$ ABJ theory	25
B.6 $U(4) \times U(2)$ ABJ theory	26

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## 1. Introduction

Low-energy limit of  $N$  coincident M2-branes on the orbifold  $\mathbb{C}^4/\mathbb{Z}_k$  is captured by the 3d  $\mathcal{N} = 6$  superconformal Chern-Simons-matter (ABJM) theory with the gauge group  $U(N)_k \times U(N)_{-k}$  [1] (see also [2]). We can take a large  $N$  limit of the ABJM theory by using t'Hooft coupling  $\lambda = N/k$  and this theory still provides fruitful developments in  $AdS_4/CFT_3$  correspondence. Meanwhile the BLG theory [3, 4] based on the Lie 3-algebra

$[X^a, X^b, X^c] = f^{abc}_d X^d$  can also lead us to another description of multiple M2-branes. If we take the structure constant  $f^{abcd}$  to be totally anti-symmetric, then the BLG theory generically has manifest  $\mathcal{N} = 8$  supersymmetry and  $SO(8)_R$  R-symmetry. In spite of such successful structures, it is known that the only nontrivial solution for a generalized Jacobi identity is the  $A_4$  algebra defined by  $f^{abcd} = \epsilon^{abcd}$  [5, 6] and the resulting  $A_4$  BLG theory can be rewritten as the  $SU(2) \times SU(2)$  ABJM theory [7]. Actually moduli space analysis of this theory [8, 9] implies that the interpretation as two indistinguishable M2-branes on  $\mathbb{C}^4/\mathbb{Z}_k$  can be possible only for  $k = 1$  and  $k = 2$ . Therefore the role of the  $A_4$  BLG theory with higher  $k$  has been somewhat unclear.

In [10], an illuminating answer has been obtained by considering  $(SU(2)_k \times SU(2)_{-k})/\mathbb{Z}_2$  rather than  $SU(2)_k \times SU(2)_{-k}$  as the correct gauge group. The authors have concluded that there are several isomorphisms even in the quantum level between

$$\boxed{U(2)_k \times U(2)_{-k} \text{ ABJM and } \mathbb{Z}_k \text{ quotient of } (SU(2)_k \times SU(2)_{-k})/\mathbb{Z}_2 \text{ BLG theory}}$$
(1.1)

where  $k$  is odd. For  $k = 2$ , isomorphism between

$$\boxed{U(2)_2 \times U(2)_{-2} \text{ ABJM and } SU(2)_2 \times SU(2)_{-2} \text{ BLG theory}}$$
(1.2)

has been also conjectured. As we will see in the next section, the additional  $\mathbb{Z}_k$  identification in (1.1) is coming from the  $U(1)_B$  baryon symmetry of  $(SU(2) \times SU(2))/\mathbb{Z}_2$  theory. More generally, they also proposed that the conjecture (1.1) can be extended to arbitrary rank  $N$  as

$$\boxed{U(N)_k \times U(N)_{-k} \text{ ABJM and } \mathbb{Z}_k \text{ quotient of } (SU(N)_k \times SU(N)_{-k})/\mathbb{Z}_N \text{ theory}}$$
(1.3)

where  $k$  and  $N$  are coprime.

The conjectures (1.1) for  $k = 1$  and (1.2) have been already tested by comparing the superconformal indices [11, 12, 13, 14] obtained by applying the localization method [15] (see also [16]) and actually nontrivial coincidences have been observed [17]. Furthermore, it has been also found in [17] that the superconformal indices of

$$\boxed{U(3)_2 \times U(2)_{-2} \text{ ABJ theory and } (SU(2)_4 \times SU(2)_{-4})/\mathbb{Z}_2 \text{ BLG theory}}$$
(1.4)

agree with each other. Thus, the nontrivial tests<sup>1</sup> beyond the moduli space analysis have been already performed for the isomorphisms between the ABJ(M) theories with the non-trivial  $\mathcal{N} = 8$  SUSY enhancements and the corresponding BLG theories<sup>2</sup>.

<sup>1</sup>In Appendix A, we provide a further evidence for the conjecture (1.2) by calculating the partition function on  $S^3$ .

<sup>2</sup>One might be curious about  $U(3)_1 \times U(2)_{-1}$  and  $U(4)_2 \times U(2)_{-2}$  ABJ theories. However it is widely believed that these theories are dual to the  $U(2)_1 \times U(2)_{-1}$  and  $U(2)_2 \times U(2)_{-2}$  ABJ theories from parity duality, respectively [2]. For a proof by considering the partition functions on  $S^3$ , see [18, 19].

In this paper, we test the conjecture (1.1) for the case without  $\mathcal{N} = 8$  SUSY enhancements by comparing their superconformal indices. We also check the conjecture (1.3) for  $N = 3$  and investigate whether extensions of the isomorphisms (1.2) and (1.4) to higher  $k$  are possible or not. This paper is organized as follows. In Section 2, we review the argument of [10] about the isomorphism (1.3). In Section 3, we briefly look at the charge quantization condition of the  $(SU(N)_k \times SU(N)_{-k})/\mathbb{Z}_N$  theories. In section 4, we describe our calculation of the superconformal indices. In Section 5, we show our results for the indices and test the conjecture (1.3). In Section 6, we investigate a possibility where the isomorphisms (1.2) and (1.4) are extended to higher  $k$ . Section 7 is devoted to conclusions and discussions.

## 2. Dual photon and moduli space of vacua

In this section, we review arguments of [10] about the conjectured isomorphism (1.3). This can be deduced from integrating out the  $U(1)$  field or comparing the classical moduli spaces.

The Lagrangian of the  $U(N)_k \times U(N)_{-k}$  ABJM theory can be expressed [10] as

$$\mathcal{L}_{u(N) \oplus u(N)} = \mathcal{L}_{su(N) \oplus su(N)}^{\text{gauged}} + \frac{Nk}{8\pi} \epsilon^{\mu\nu\lambda} B_\mu H_{\nu\lambda}, \quad (2.1)$$

where  $B_\mu$  is the gauge field of the  $U(1)_B$  baryon symmetry.  $H_{\mu\nu}$  is the field strength of the trivial  $U(1)$ , which does not couple to all the fields in the  $SU(N)_k \times SU(N)_{-k}$  theory. The second term is the so-called BF term, which is required to make the theory invariant under the  $\mathcal{N} = 6$  supersymmetry after gauging the  $U(1)_B$  symmetry. Introducing the Lagrange multiplier  $\sigma$  leads to

$$\mathcal{L}_{u(N) \oplus u(N)} = \mathcal{L}_{su(N) \oplus su(N)}^{\text{gauged}} + \frac{Nk}{8\pi} \epsilon^{\mu\nu\lambda} B_\mu H_{\nu\lambda} + \frac{N}{8\pi} \sigma \epsilon^{\mu\nu\lambda} \partial_\mu H_{\nu\lambda}. \quad (2.2)$$

Integrating this by parts, we obtain

$$\mathcal{L}_{u(N) \oplus u(N)} = \mathcal{L}_{su(N) \oplus su(N)}^{\text{gauged}} + \frac{Nk}{8\pi} \epsilon^{\mu\nu\lambda} B_\mu H_{\nu\lambda} - \frac{N}{8\pi} \epsilon^{\mu\nu\lambda} \partial_\mu \sigma H_{\nu\lambda}. \quad (2.3)$$

Then, the equation of motion for  $H_{\mu\nu}$  is

$$B_\mu = \frac{1}{k} \partial_\mu \sigma. \quad (2.4)$$

From this equation, we find

$$\mathcal{L}_{u(N) \oplus u(N)}(Z^A, \psi_A, B_\mu, H_{\mu\nu\lambda}) = \mathcal{L}_{su(N) \oplus su(N)}(e^{\frac{i}{k}\sigma} Z^A, e^{\frac{i}{k}\sigma} \psi^A), \quad (A = 1, \dots, 4) \quad (2.5)$$

where the  $U(1)_B$  gauge transformation  $B_\mu \rightarrow B_\mu + \partial_\mu \theta$  in the language of  $\sigma$  is given by

$$\sigma \rightarrow \sigma + k\theta. \quad (2.6)$$

The last term of (2.2) implies that the periodicity of  $\sigma$  is determined by the charge quantization condition of  $H_{\mu\nu}$ . Note that the charge quantization condition is different

from the usual Dirac quantization condition since  $U(N)$  is not just a product of  $U(1)$  and  $SU(N)$  but rather it is  $(U(1) \times SU(N))/\mathbb{Z}_N$ . Recall that  $H$  is a sum of a field strength of each  $U(1)$  factor of  $U(N) \times U(N)$  gauge group. Finally the condition is given by

$$\int dH = \int \frac{1}{2} \epsilon^{\mu\nu\lambda} \partial_\mu H_{\nu\lambda} \in \frac{4\pi}{N} \mathbb{Z}, \quad (2.7)$$

which leads the periodicity of  $\sigma$  to  $2\pi$ . Thus, we must impose the following identification on the fields

$$\hat{Z}^A \sim e^{\frac{2\pi i}{k}} \hat{Z}^A, \quad \hat{\psi}^A \sim e^{\frac{2\pi i}{k}} \hat{\psi}^A. \quad (2.8)$$

where we define  $\hat{Z}^A$  and  $\hat{\psi}_A$  as  $\hat{Z}^A = e^{\frac{i}{k}\sigma} Z^A$  and  $\hat{\psi}_A = e^{\frac{i}{k}\sigma} \psi_A$ , respectively. From this fact, the authors of [10] have concluded that the  $U(N)_k \times U(N)_{-k}$  ABJM theory is also equivalent to a  $\mathbb{Z}_k$  identification on the  $(SU(N)_k \times SU(N)_{-k})/\mathbb{Z}_N$  theory. As we will see later, this equivalence can hold if we impose an additional constraint on  $N$  and  $k$ .

Next we consider the moduli space of the  $(SU(2)_k \times SU(2)_{-k})/\mathbb{Z}_2$  theory with the  $\mathbb{Z}_k$  identification and check the above result. Discussion for generalization to arbitrary rank  $N$  is essentially the same [10]. Setting the scalar potential to be zero, we can take  $Z^A$  up to gauge transformation as

$$Z^A = \frac{1}{\sqrt{2}} r_1^A - \frac{i}{\sqrt{2}} r_2^A \sigma_3, \quad (2.9)$$

where  $r_1^A$  and  $r_2^A$  are complex numbers. These can be regarded as the center of mass coordinate and the relative coordinate of two M2-branes, respectively. For a later convenience, we take

$$r_1^A = \frac{1}{2}(z_1^A + z_2^A), \quad r_2^A = \frac{i}{2}(z_1^A - z_2^A). \quad (2.10)$$

Recall that the moduli space of  $SU(2) \times SU(2)$  theory is shown to be the orbifold  $(\mathbb{C}^4 \times \mathbb{C}^4)/D_{2k}$  [8, 9]. Here  $D_n$  is the dihedral group of order  $2n$ , which is equivalent to the semi-direct product of  $\mathbb{Z}_n$  and  $\mathbb{Z}_2$  with  $\mathbb{Z}_2$  acting on  $\mathbb{Z}_n$  by inversion. Except for the modification (2.7) of the charge quantization condition and the  $\mathbb{Z}_k$  identification (2.8), the same argument holds also in the present case. Thus, we can show that the moduli space of the  $(SU(2) \times SU(2))/\mathbb{Z}_2$  theory with the  $\mathbb{Z}_k$  identification is given by

$$\begin{aligned} \mathbb{Z}_k^{U(1)} &: & z_1^A &\sim e^{\frac{2\pi i}{k}} z_1^A, & z_2^A &\sim e^{\frac{2\pi i}{k}} z_2^A, \\ \mathbb{Z}_2^{\text{perm}} &: & z_1^A &\sim z_2^A, \\ \mathbb{Z}_k^{(SU(2) \times SU(2))/\mathbb{Z}_2} &: & z_1^A &\sim e^{\frac{2\pi i}{k}} z_1^A, & z_2^A &\sim e^{-\frac{2\pi i}{k}} z_2^A, \end{aligned} \quad (2.11)$$

where  $\mathbb{Z}_k^{U(1)}$  is the  $\mathbb{Z}_k$  identification (2.8) imposed to  $z_1^A, z_2^A$  and  $\mathbb{Z}_2^{\text{perm}}$  is a permutation of two indistinguishable M2-branes. Note that the last one is slightly different from the one of the  $SU(2) \times SU(2)$  theory given by

$$\mathbb{Z}_{2k}^{SU(2) \times SU(2)} : \quad z_1^A \sim e^{\frac{\pi i}{k}} z_1^A, \quad z_2^A \sim e^{-\frac{\pi i}{k}} z_2^A. \quad (2.12)$$

The difference is comes from the modified charge quantization condition (2.7). As we mentioned above,  $\mathbb{Z}_2^{\text{perm}}$  and  $\mathbb{Z}_k^{(SU(2) \times SU(2))/\mathbb{Z}_2}$  identifications generate the dihedral group  $D_k$ .

The moduli space of  $U(2) \times U(2)$  ABJM theory is

$$\frac{(\mathbb{R}^8/\mathbb{Z}_k) \times (\mathbb{R}^8/\mathbb{Z}_k)}{\mathbb{Z}_2}, \quad (2.13)$$

which is given by the following quotient

$$\begin{aligned} \mathbb{Z}_k^{(1)} : \quad & z_1^A \sim e^{\frac{2\pi i}{k}} z_1^A, \quad z_2^A \sim z_2^A, \\ \mathbb{Z}_2^{\text{perm}} : \quad & z_1^A \sim z_2^A, \\ \mathbb{Z}_k^{(2)} : \quad & z_1^A \sim z_1^A, \quad z_2^A \sim e^{-\frac{2\pi i}{k}} z_2^A. \end{aligned} \quad (2.14)$$

Here  $\mathbb{Z}_k^{(1)}$  and  $\mathbb{Z}_k^{(2)}$  are the  $\mathbb{Z}_k$  identifications of each M2-brane. We can easily show that (2.11) and (2.14) are equal with each other if and only if  $k$  is odd<sup>3</sup>. For arbitrary  $N$ , similar discussion leads us to the conclusion that the  $U(N)_k \times U(N)_{-k}$  ABJM theory is isomorphic to the  $\mathbb{Z}_k$  quotient of the  $(SU(N)_k \times SU(N)_{-k})/\mathbb{Z}_N$  theory if  $N$  and  $k$  are coprime [10].

### 3. Magnetic charge and charge quantization condition

Here we briefly look at the charge quantization condition of the  $(SU(N)_k \times SU(N)_{-k})/\mathbb{Z}_N$  theories. The full global symmetry of the  $U(N_1) \times U(N_2)$  ABJ(M) theory is  $SO(6)_R \times U(1)_T$ . Here  $U(1)_T$  is the topological symmetry of the ABJ(M) theory whose conserved current is given by<sup>4</sup>

$$J^\mu = \frac{1}{16\pi} \epsilon^{\mu\nu\rho} \left( \text{Tr} F_{\nu\rho} + \text{Tr} \tilde{F}_{\nu\rho} \right). \quad (3.1)$$

$F$  and  $\tilde{F}$  are the field strengths of  $U(N_1)$  and  $U(N_2)$  gauge fields, respectively. The operators carrying the  $U(1)_T$  charge are called monopole operators [20, 21] and involve a non-zero magnetic flux in the diagonal  $U(1)$  gauge group. The monopole operators can be labeled by the GNO charges  $n_1, \dots, n_{N_1}$  and  $\tilde{n}_1, \dots, \tilde{n}_{N_2}$  which are the monopole charges for the Cartan part of the gauge group  $U(N_1) \times U(N_2)$  [22]. The GNO charges label the magnetic flux on  $S^2$  surrounding the insertion point of the operator and their summation corresponds to the  $U(1)_T$  charge as

$$Q_T = \frac{k}{4} \left( \sum_{i=1}^{N_1} n_i + \sum_{a=1}^{N_2} \tilde{n}_a \right). \quad (3.2)$$

The equations of motion for the gauge fields set  $\text{Tr} F - \text{Tr} \tilde{F} = 0$  and therefore  $\sum_i n_i = \sum_a \tilde{n}_a$ . Thus, the  $U(1)_T$  charge can be expressed as

$$Q_T = \frac{k}{2} T \quad \text{with} \quad T = \sum_{i=1}^{N_1} n_i = \sum_{a=1}^{N_2} \tilde{n}_a. \quad (3.3)$$

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<sup>3</sup>If we parametrize  $k = 2l - 1$ ,  $(\mathbb{Z}_k^{U(1)} \times \mathbb{Z}_k^{(SU(2) \times SU(2))/\mathbb{Z}_2})^l$  and  $((\mathbb{Z}_k^{U(1)})^{-1} \times \mathbb{Z}_k^{(SU(2) \times SU(2))/\mathbb{Z}_2})^l$  are equal to  $\mathbb{Z}_k^{(1)}$  and  $\mathbb{Z}_k^{(2)}$ , respectively.

<sup>4</sup>We use the same notation as in [17].

Let us denote  $w_i$  ( $i = 1, \dots, \text{rank}(G)$ ) as the weight vector of the gauge group  $G$  in an irreducible representation and  $\Lambda(G)$  as the weight lattice. The quantization condition imposes  $\exp(i e \sum_i n_i w_i) = 1$  and this implies that

$$e \sum_i n_i w_i = 2\pi\mathbb{Z} \quad (3.4)$$

for all  $w \in \Lambda(G)$ <sup>5</sup>. Here we consider the  $SO(3) = SU(2)/\mathbb{Z}_2$  and  $SU(2)$  as a simple example of dual groups<sup>6</sup>. In this case, we have

$$\begin{aligned} \Lambda(SO(3)) &= \{0, \pm 1, \pm 2, \pm 3, \dots\}, \\ \Lambda(SU(2)) &= \{0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \dots\}. \end{aligned} \quad (3.5)$$

As a result, we find that the magnetic charges must satisfy

$$\begin{aligned} e n_i &= 2\pi\mathbb{Z} & \text{for } SO(3), \\ e n_i &= 4\pi\mathbb{Z} & \text{for } SU(2). \end{aligned} \quad (3.6)$$

In the  $A_4$  BLG theory, we have  $(SU(2) \times SU(2))/\mathbb{Z}_2$  gauge group, where the  $\mathbb{Z}_2$  is embedded diagonally in the product of the centers of the two  $SU(2)$  factors. Note that this is indistinguishable from  $(SU(2)/\mathbb{Z})^2$ . Therefore, the GNO charges for  $(SU(2) \times SU(2))/\mathbb{Z}_2$  gauge group are allowed to be half the value of those of  $SU(2) \times SU(2)$  gauge group. In our notation this implies that

$$\begin{aligned} n_i &= \frac{1}{2}\mathbb{Z} & \text{for } (SU(2) \times SU(2))/\mathbb{Z}_2, \\ n_i &= \mathbb{Z} & \text{for } SU(2) \times SU(2). \end{aligned} \quad (3.7)$$

A similar discussion can be applied to the  $(SU(3) \times SU(3))/\mathbb{Z}_3$  theories. In this case, we finally obtain

$$\begin{aligned} n_i &= \frac{1}{3}\mathbb{Z} & \text{for } (SU(3) \times SU(3))/\mathbb{Z}_3, \\ n_i &= \mathbb{Z} & \text{for } SU(3) \times SU(3). \end{aligned} \quad (3.8)$$

As we will see in the next section, the superconformal index is given by summation over contributions from each GNO charge. Therefore we have to take account of the difference of the charge quantization conditions for calculating the index.

#### 4. The superconformal index

In this section, we derive useful expressions for the superconformal indices of the  $U(N_1)_k \times U(N_2)_{-k}$  ABJ(M) and  $SU(N)_k \times SU(N)_{-k}$  theories including the BLG theory. The superconformal index is defined by

$$I(x, z) = \text{Tr} \left[ (-1)^F x^{\epsilon + j_3} z^h \right], \quad (4.1)$$

---

<sup>5</sup>This also means that  $e\vec{n}$  corresponds to the dual lattice  $\Lambda^*(G)$ . This is a weight lattice of a magnetic (or Langlands) dual group  $G^\vee$  and  $e\vec{n}$  is its weight vector.

<sup>6</sup>For a general  $SU(N)$  group, the dual relation is given by  $SU(N)^\vee = SU(N)/\mathbb{Z}_N$ .

where  $F$ ,  $\epsilon$ ,  $j_3$  and  $h$  are the fermion number, the energy (or equivalently the conformal dimension), the projection of spin and the charge of a flavor symmetry, respectively. This quantity is a powerful tool for distinguishing theories with a same moduli space.

On the ABJ(M) side, we consider the index with the fixed topological charge  $T$ . If there exists an isomorphism as the conjectures (1.1)-(1.4), the index of the  $SU(N) \times SU(N)$  theory must have contribution from charges of certain symmetry corresponding to  $U(1)_T$  symmetry. As noted in [23], we can write the BLG theory of the product gauge group formulation [7] in  $\mathcal{N} = 2$  superspace. Of the original  $SO(8)_R$  R-symmetry this formulation manifestly remains only the subgroup  $SU(4) \times U(1)_R$ . Because this  $U(1)_R$  is not related to the baryonic symmetry, this has nothing to do with the  $U(1)_T$  symmetry in the ABJ(M) theory. Thus the topological charge  $T$  of the ABJ(M) theory should correspond to the charge of a  $U(1)$  subgroup of the  $SU(4) \sim SO(6)$  as discussed in [17]. We denote this  $U(1)$  subgroup as  $U(1)_t$ . On the BLG side, we treat this  $U(1)_t$  as the flavor symmetry whose charge assignments are  $+1(-1)$  to the (anti-)bi-fundamental. Therefore we introduce the variable  $z$  to distinguish the  $U(1)_t$  symmetry of the  $SU(N) \times SU(N)$  theory and compare the index with the one of the ABJ(M) theory.

#### 4.1 $U(N_1)_k \times U(N_2)_{-k}$ ABJ theory

By applying the localization method [15, 16], the (whole) superconformal index of the  $U(N_1)_k \times U(N_2)_{-k}$  ABJ theory with  $z = 1$  can be represented as

$$I_{\text{ABJ}}(x) = \sum_{\{n\}, \{\tilde{n}\}} \frac{x^{\epsilon_0}}{(\text{sym})} \int_{-\pi}^{\pi} \frac{d^{N_1} \lambda}{(2\pi)^{N_1}} \frac{d^{N_2} \tilde{\lambda}}{(2\pi)^{N_2}} e^{S_0} \exp \left[ \sum_{p=1}^{\infty} \frac{1}{p} f_{\text{tot}}(x^p, e^{ip\lambda}, e^{ip\tilde{\lambda}}) \right], \quad (4.2)$$

where  $n_i$  and  $\tilde{n}_a$  are the GNO charges,  $\lambda$  and  $\tilde{\lambda}$  are constant holonomy zero modes and

$$\begin{aligned} f_{\text{tot}} &= f_{\text{vec}} + f_{\text{hyper}}, \\ f_{\text{vec}}(x, e^{i\lambda}, e^{i\tilde{\lambda}}) &= - \sum_{i \neq j}^{N_1} \left( e^{i(\lambda_i - \lambda_j)} x^{|n_i - n_j|} \right) - \sum_{a \neq b}^{N_2} \left( e^{i(\tilde{\lambda}_a - \tilde{\lambda}_b)} x^{|\tilde{n}_a - \tilde{n}_b|} \right), \\ f_{\text{hyper}}(x, e^{i\lambda}, e^{i\tilde{\lambda}}) &= 2 \sum_{i,b} \left( \frac{x^{1/2}}{1+x} x^{|n_i - \tilde{n}_b|} e^{i(\lambda_i - \tilde{\lambda}_b)} + \frac{x^{1/2}}{1+x} x^{|n_i - \tilde{n}_b|} e^{-i(\lambda_i - \tilde{\lambda}_b)} \right), \\ S_0 &= ik \sum_i n_i \lambda_i - ik \sum_a \tilde{n}_a \tilde{\lambda}_a, \\ \epsilon_0 &= \sum_{i,b} |n_i - \tilde{n}_b| - \frac{1}{2} \sum_{i,j} |n_i - n_j| - \frac{1}{2} \sum_{a,b} |\tilde{n}_a - \tilde{n}_b|. \end{aligned} \quad (4.3)$$

The factor “(sym)” denotes the rank of Weyl group for unbroken gauge group. By using the formula

$$\sum_{p=1}^{\infty} \frac{1}{p} e^{ip\lambda} z^p = -\log(1 - ze^{i\lambda}),$$

the contribution from the vector multiplet is rewritten as

$$\exp \left[ \sum_{p=1}^{\infty} \frac{1}{p} f_{\text{vec}}(x^p, e^{ip\lambda}, e^{ip\tilde{\lambda}}) \right] = \prod_{i \neq j} \left( 1 - x^{|n_i - n_j|} e^{i(\lambda_i - \lambda_j)} \right) \prod_{a \neq b} \left( 1 - x^{|\tilde{n}_a - \tilde{n}_b|} e^{i(\tilde{\lambda}_a - \tilde{\lambda}_b)} \right).$$



$$(4.4)$$

The contribution from the hyper multiplet is a bit more complicated. By using

$$\sum_{p=1}^{\infty} \frac{1}{p} \frac{y^p}{1+x^p} e^{ip\lambda} = - \sum_{m=0}^{\infty} \log \left( 1 - yx^{2m} e^{i\lambda} \right) + \sum_{m=0}^{\infty} \log \left( 1 - yx^{2m+1} e^{i\lambda} \right), \quad (4.5)$$

we obtain

$$\begin{aligned} & \exp \left[ \sum_{p=1}^{\infty} \frac{1}{p} f_{\text{hyper}}(x^p, e^{ip\lambda}, e^{ip\tilde{\lambda}}) \right] \\ &= \prod_{i,b} \left[ \prod_{m=0}^{\infty} \frac{1 - x^{2m+3/2+|n_i-\tilde{n}_b|} e^{i(\lambda_i-\tilde{\lambda}_b)}}{1 - x^{2m+1/2+|n_i-\tilde{n}_b|} e^{i(\lambda_i-\tilde{\lambda}_b)}} \frac{1 - x^{2m+3/2+|n_i-\tilde{n}_b|} e^{-i(\lambda_i-\tilde{\lambda}_b)}}{1 - x^{2m+1/2+|n_i-\tilde{n}_b|} e^{-i(\lambda_i-\tilde{\lambda}_b)}} \right]^2 \\ &= \prod_{i,b} \left[ \frac{(x^{3/2+|n_i-\tilde{n}_b|} e^{i(\lambda_i-\tilde{\lambda}_b)}; x^2)_{\infty}}{(x^{1/2+|n_i-\tilde{n}_b|} e^{i(\lambda_i-\tilde{\lambda}_b)}; x^2)_{\infty}} \frac{(x^{3/2+|n_i-\tilde{n}_b|} e^{-i(\lambda_i-\tilde{\lambda}_b)}; x^2)_{\infty}}{(x^{1/2+|n_i-\tilde{n}_b|} e^{-i(\lambda_i-\tilde{\lambda}_b)}; x^2)_{\infty}} \right]^2 \\ &= \prod_{i,b} \mathcal{I}_{\text{hyper}} \left( x, n_i - \tilde{n}_b, e^{i(\lambda_i-\tilde{\lambda}_b)} \right), \end{aligned} \quad (4.6)$$

where  $(a; q)_{\infty} = \prod_{m=0}^{\infty} (1 - aq^m)$  is the q-Pochhammer symbol and

$$\mathcal{I}_{\text{hyper}}(x, n, y) = \left[ \frac{(x^{3/2+|n|} y; x^2)_{\infty}}{(x^{1/2+|n|} y; x^2)_{\infty}} \frac{(x^{3/2+|n|} y^{-1}; x^2)_{\infty}}{(x^{1/2+|n|} y^{-1}; x^2)_{\infty}} \right]^2 \quad (4.7)$$

Thus, the superconformal index becomes the following simple form

$$\begin{aligned} I_{\text{ABJ}}(x) &= \sum_{\{n\}, \{\tilde{n}\}} \frac{x^{\epsilon_0}}{(\text{sym})} \int_{-\pi}^{\pi} \frac{d^{N_1} \lambda}{(2\pi)^{N_1}} \frac{d^{N_2} \tilde{\lambda}}{(2\pi)^{N_2}} e^{S_0} \\ &\quad \times \prod_{i \neq j} \left( 1 - x^{|n_i - n_j|} e^{i(\lambda_i - \lambda_j)} \right) \prod_{a \neq b} \left( 1 - x^{|\tilde{n}_a - \tilde{n}_b|} e^{i(\tilde{\lambda}_a - \tilde{\lambda}_b)} \right) \\ &\quad \times \prod_{i,b} \mathcal{I}_{\text{hyper}} \left( x, n_i - \tilde{n}_b, e^{i(\lambda_i - \tilde{\lambda}_b)} \right). \end{aligned} \quad (4.8)$$

Next we perform a change of variables as

$$\mu_i = \lambda_i - \lambda_{N_1} \quad (i = 1, \dots, N_1 - 1), \quad \mu_{N_1} = \lambda_{N_1}, \quad \nu_a = \tilde{\lambda}_a - \lambda_{N_1}. \quad (4.9)$$

Then we can easily integrate over  $\mu_{N_1}$  and obtain

$$\begin{aligned} I_{\text{ABJ}}(x) &= \sum_{n_1, \dots, n_{N_1-1}, \tilde{n}_1, \dots, \tilde{n}_{N_2}} \frac{x^{\epsilon'_0}}{(\text{sym})} \int \frac{d^{N_1-1} \mu}{(2\pi)^{N_1-1}} \frac{d^{N_2} \nu}{(2\pi)^{N_2}} e^{S'_0} \\ &\quad \times \prod_{i \neq j} \left( 1 - x^{|n_i - n_j|} e^{i(\mu_i - \mu_j)} \right) \prod_{a \neq b} \left( 1 - x^{|\tilde{n}_a - \tilde{n}_b|} e^{i(\nu_a - \nu_b)} \right) \\ &\quad \times \prod_i \left( 1 - x^{|n_i + \sum_j n_j - \sum_a \tilde{n}_a|} e^{i\mu_i} \right) \left( 1 - x^{|n_i + \sum_j n_j - \sum_a \tilde{n}_a|} e^{-i\mu_i} \right) \end{aligned}$$

$$\times \prod_{i,b} \mathcal{I}_{\text{hyper}} \left( x, n_i - \tilde{n}_b, e^{i(\mu_i - \nu_b)} \right) \prod_a \mathcal{I}_{\text{hyper}} \left( x, \sum_b \tilde{n}_b - \tilde{n}_a - \sum_i n_i, e^{i\nu_a} \right), \quad (4.10)$$

where

$$\begin{aligned} S'_0 &= ik \sum_{i=1}^{N_1-1} n_i \mu_i - ik \sum_a \tilde{n}_a \nu_a, \\ \epsilon'_0 &= \sum_{i,b} |n_i - \tilde{n}_b| - \sum_{i < j} |n_i - n_j| - \sum_{a < b} |\tilde{n}_a - \tilde{n}_b| \\ &\quad + \sum_a \left| -\sum_j n_j + \sum_b \tilde{n}_b - \tilde{n}_a \right| - \sum_i \left| n_i + \sum_j n_j - \sum_b \tilde{n}_b \right|. \end{aligned} \quad (4.11)$$

Furthermore, we perform a change of the variables as

$$y_i = e^{i\mu_i}, \quad w_a = e^{i\nu_a}, \quad (4.12)$$

where each of them run over the unit circle in the complex plane. Then we obtain

$$\begin{aligned} I_{\text{ABJ}}(x) &= \sum_{n_1, \dots, n_{N_1-1}, \tilde{n}_1, \dots, \tilde{n}_{N_2}} \frac{x^{\epsilon'_0}}{(\text{sym})} \oint \frac{d^{N_1-1}y}{(2\pi i)^{N_1-1}} \frac{d^{N_2}w}{(2\pi i)^{N_2}} \prod_i \left( y_i^{kn_i-1} \right) \prod_a \left( w_i^{-k\tilde{n}_a-1} \right) \\ &\quad \times \prod_{i \neq j} \left( 1 - x^{|n_i - n_j|} y_i y_j^{-1} \right) \prod_{a \neq b} \left( 1 - x^{|\tilde{n}_a - \tilde{n}_b|} w_a w_b^{-1} \right) \\ &\quad \times \prod_i \left( 1 - x^{|n_i + \sum_j n_j - \sum_a \tilde{n}_a|} y_i \right) \left( 1 - x^{|n_i + \sum_j n_j - \sum_a \tilde{n}_a|} y_i^{-1} \right) \\ &\quad \times \prod_{i,b} \mathcal{I}_{\text{hyper}} \left( x, n_i - \tilde{n}_b, y_i w_b^{-1} \right) \prod_a \mathcal{I}_{\text{hyper}} \left( x, \sum_b \tilde{n}_b - \tilde{n}_a - \sum_i n_i, w_a \right). \end{aligned} \quad (4.13)$$

Thus we can write the index with the fixed topological charge  $T$  as

$$\begin{aligned} I_{\text{ABJ}}^{(T)}(x) &= \sum_{n_1, \dots, n_{N_1-1}, \tilde{n}_1, \dots, \tilde{n}_{N_2}} \delta_{\sum_a \tilde{n}_a, T} \frac{x^{\epsilon'_0}}{(\text{sym})} \\ &\quad \oint \frac{d^{N_1-1}y}{(2\pi i)^{N_1-1}} \frac{d^{N_2}w}{(2\pi i)^{N_2}} \prod_i \left( y_i^{kn_i-1} \right) \prod_a \left( w_i^{-k\tilde{n}_a-1} \right) \\ &\quad \times \prod_{i \neq j} \left( 1 - x^{|n_i - n_j|} y_i y_j^{-1} \right) \prod_{a \neq b} \left( 1 - x^{|\tilde{n}_a - \tilde{n}_b|} w_a w_b^{-1} \right) \\ &\quad \times \prod_i \left( 1 - x^{|n_i + \sum_j n_j - \sum_a \tilde{n}_a|} y_i \right) \left( 1 - x^{|n_i + \sum_j n_j - \sum_a \tilde{n}_a|} y_i^{-1} \right) \\ &\quad \times \prod_{i,b} \mathcal{I}_{\text{hyper}} \left( x, n_i - \tilde{n}_b, y_i w_b^{-1} \right) \prod_a \mathcal{I}_{\text{hyper}} \left( x, \sum_b \tilde{n}_b - \tilde{n}_a - \sum_i n_i, w_a \right). \end{aligned}$$

(4.14)

The integration can be performed by expanding the integrand as power series of  $y_i, w_a$  and picking up the poles at the origin.

## 4.2 $SU(N)_k \times SU(N)_{-k}$ theory

Let us consider the  $SU(N)_k \times SU(N)_{-k}$  theory or the  $(SU(N)_k \times SU(N)_{-k})/\mathbb{Z}_N$  theory including the BLG theory. The difference of global structure of the gauge group only affects the value of the GNO charges. As we mentioned above, we treat the  $U(1)_t$  symmetry as the flavor symmetry which assigns the flavor charges  $+1$  and  $-1$  to the bi-fundamental and anti-bi-fundamental multiplets, respectively. Then, the superconformal index of the  $SU(N)_k \times SU(N)_{-k}$  theory is given by

$$\begin{aligned}
& I_{\text{BLG}}(x, z) \\
&= \sum_{\{n\}, \{\tilde{n}\}} \delta_{\sum_i n_i, 0} \delta_{\sum_i \tilde{n}_i, 0} \frac{x^{\epsilon_0}}{(\text{sym})} \int \frac{d^N \lambda}{(2\pi)^N} \frac{d^N \tilde{\lambda}}{(2\pi)^N} \delta\left(\sum_i \lambda_i\right) \delta\left(\sum_i \tilde{\lambda}_i\right) e^{S_0} \\
&\quad \times \prod_{i \neq j} \left[ \left(1 - x^{|n_i - n_j|} e^{i(\lambda_i - \lambda_j)}\right) \left(1 - x^{|\tilde{n}_i - \tilde{n}_j|} e^{i(\tilde{\lambda}_i - \tilde{\lambda}_j)}\right) \right] \\
&\quad \times \prod_{i, j} \mathcal{I}_{\text{hyper}} \left( x, n_i - \tilde{n}_j, z e^{i(\lambda_i - \tilde{\lambda}_j)} \right) \\
&= \sum_{\{n\}, \{\tilde{n}\}} \delta_{\sum_i n_i, 0} \delta_{\sum_i \tilde{n}_i, 0} \frac{x^{\epsilon_0}}{(\text{sym})} \int \frac{d^{N-1} \lambda}{(2\pi)^{N-1}} \frac{d^{N-1} \tilde{\lambda}}{(2\pi)^{N-1}} e^{S_0} \\
&\quad \times \prod_{i \neq j} \left[ \left(1 - x^{|n_i - n_j|} e^{i(\lambda_i - \lambda_j)}\right) \left(1 - x^{|\tilde{n}_i - \tilde{n}_j|} e^{i(\tilde{\lambda}_i - \tilde{\lambda}_j)}\right) \right] \\
&\quad \times \prod_{i, j} \mathcal{I}_{\text{hyper}} \left( x, n_i - \tilde{n}_j, z e^{i(\lambda_i - \tilde{\lambda}_j)} \right) \Bigg|_{\lambda_N = -\sum_{i=1}^{N-1} \lambda_i, \tilde{\lambda}_N = -\sum_{i=1}^{N-1} \tilde{\lambda}_i} \\
&= \sum_{\{n\}, \{\tilde{n}\}} \delta_{\sum_i n_i, 0} \delta_{\sum_i \tilde{n}_i, 0} \frac{x^{\epsilon_0}}{(\text{sym})} \oint \frac{d^{N-1} y}{(2\pi i)^{N-1}} \frac{d^{N-1} w}{(2\pi i)^{N-1}} \prod_{i=1}^{N-1} (y_i w_i)^{-1} e^{S_0} \\
&\quad \times \prod_{i \neq j} \left[ \left(1 - x^{|n_i - n_j|} y_i y_j^{-1}\right) \left(1 - x^{|\tilde{n}_i - \tilde{n}_j|} w_i w_j^{-1}\right) \right] \\
&\quad \times \prod_{i, j} \mathcal{I}_{\text{hyper}} \left( x, n_i - \tilde{n}_j, z y_i w_j^{-1} \right) \Bigg|_{y_N = \prod_{i=1}^{N-1} y_i^{-1} w_N = \prod_{i=1}^{N-1} w_i^{-1}}. \tag{4.15}
\end{aligned}$$

Similarly to the ABJ case, the integration can be also performed by expanding the integrand as power series of  $y_i, w_i$  and picking up the poles at the origin.

## 5. Test of the conjectured isomorphisms

In this section, we present our result of the superconformal indices for the  $U(N) \times U(N)$  ABJM theory and the  $(SU(N) \times SU(N))/\mathbb{Z}_N$  theory including the BLG theory. By using

our formula (4.14) and (4.15), we compute the indices and test the conjectured isomorphisms (1.3) for  $N = 2$  and  $N = 3$ .

### 5.1 $N = 2$

Here we consider the  $U(2)_k \times U(2)_{-k}$  ABJM theory and the  $\mathbb{Z}_k$  quotient of the  $(SU(2)_k \times SU(2)_{-k})/\mathbb{Z}_2$  BLG theory. As we mentioned in Section 2, we must take  $k$  to be odd in order to match the moduli spaces of the both theories. We compute the superconformal indices up to the fifth orders in  $x$ .

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 24x^3 + 44x^4$
$ 0, 0\rangle 0, 0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$
$ 1, -1\rangle 1, -1\rangle$	$16x^3 + 32x^4$
$T = 1$	$4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}}$
$ 1, 0\rangle 1, 0\rangle$	$4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}}$
$T = 2$	$17x^3 + 48x^4$
$ 1, 1\rangle 1, 1\rangle$	$10x^3 + 16x^4$
$ 2, 0\rangle 2, 0\rangle$	$7x^3 + 32x^4$

Table 1: The superconformal index of the  $U(2)_3 \times U(2)_{-3}$  ABJM theory up to  $\mathcal{O}(x^5)$ . A symbol  $|n_1, n_2\rangle|\tilde{n}_1, \tilde{n}_2\rangle$  denotes the contribution from the GNO charges  $(n_1, n_2, \tilde{n}_1, \tilde{n}_2)$ .  $T$  represents the topological charge.

Let us compare the ABJM index in an individual topological charge  $T$  with the BLG index in a particular monomial of  $z$ . First, we consider the case for  $k = 3$ . In Table 1, we show the contributions from each GNO charge to the index in the  $U(2)_3 \times U(2)_{-3}$  ABJM theory. To summarize the result, the ABJM indices with the fixed topological charge  $T$  are given by<sup>7</sup>

$$\begin{aligned}
I_{\text{ABJM}, k=3}^{(T=0)}(x) &= 1 + 4x + 12x^2 + 24x^3 + 44x^4, \\
I_{\text{ABJM}, k=3}^{(T=1)}(x) &= 4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}}, \\
I_{\text{ABJM}, k=3}^{(T=2)}(x) &= 17x^3 + 48x^4,
\end{aligned} \tag{5.1}$$

up to  $\mathcal{O}(x^5)$ . The result of the  $(SU(2)_3 \times SU(2)_{-3})/\mathbb{Z}_2$  BLG theory is shown in Table 2. Note that we have to sum over all relevant GNO charges on the BLG side in order to obtain all the contributions to the fixed charge  $U(1)_t$ . The BLG index is summarized as

$$\begin{aligned}
I_{\text{BLG}, k=3}(x, z) &= 1 + 4x + 12x^2 + 24x^3 + 44x^4 \\
&\quad + z(6x^{\frac{3}{2}} + 22x^{\frac{5}{2}} + 12x^{\frac{7}{2}}) + z^{-1}(6x^{\frac{3}{2}} + 22x^{\frac{5}{2}} + 12x^{\frac{7}{2}})
\end{aligned}$$

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<sup>7</sup>Here we explicitly show the results only for non-negative  $T$  since  $I_{\text{ABJM}}^{(-T)}(x) = I_{\text{ABJM}}^{(T)}(x)$ .

GNO charges	Index contribution
$ 0, 0\rangle 0, 0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$ $+ z^2(3x + 8x^2 + 12x^3 + 8x^4) + z^{-2}(3x + 8x^2 + 12x^3 + 8x^4)$ $+ z^4(6x^2 + 12x^3 + 12x^4) + z^{-4}(6x^2 + 12x^3 + 12x^4)$ $+ z^6(10x^3 + 16x^4) + z^{-6}(10x^3 + 16x^4) + 15z^8x^4 + 15z^{-8}x^4$
$ 1/2, -1/2\rangle 1/2, -1/2\rangle$	$z(6x^{\frac{3}{2}} + 22x^{\frac{5}{2}} + 12x^{\frac{7}{2}}) + z^{-1}(6x^{\frac{3}{2}} + 22x^{\frac{5}{2}} + 12x^{\frac{7}{2}})$ $+ z^3(4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}}) + z^{-3}(4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}})$ $+ z^5(10x^{\frac{5}{2}} + 28x^{\frac{7}{2}}) + z^{-5}(10x^{\frac{5}{2}} + 28x^{\frac{7}{2}}) + 18z^7x^{\frac{7}{2}} + 18z^{-7}x^{\frac{7}{2}}$
$ 1, -1\rangle 1, -1\rangle$	$16x^3 + 32x^4$ $+ z^2(15x^3 + 32x^4) + z^{-2}(15x^3 + 32x^4)$ $+ z^4(12x^3 + 32x^4) + z^{-4}(12x^3 + 32x^4)$ $+ z^6(7x^3 + 32x^4) + z^{-6}(7x^3 + 32x^4) + 16z^8x^4 + 16z^{-8}x^4$

Table 2: The superconformal index of the  $(SU(2)_3 \times SU(2)_{-3})/\mathbb{Z}_2$  BLG theory up to  $\mathcal{O}(x^5)$ . If we take the additional  $\mathbb{Z}_3$  quotient, only the terms whose powers of  $z$  is multiples of 3 remain.

$$\begin{aligned}
& + z^2(3x + 8x^2 + 27x^3 + 40x^4) + z^{-2}(3x + 8x^2 + 27x^3 + 40x^4) \\
& + z^3(4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}}) + z^{-3}(4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}}) \\
& + z^4(6x^2 + 24x^3 + 44x^4) + z^{-4}(6x^2 + 24x^3 + 44x^4) \\
& + z^5(10x^{\frac{5}{2}} + 28x^{\frac{7}{2}}) + z^{-5}(10x^{\frac{5}{2}} + 28x^{\frac{7}{2}}) \\
& + z^6(17x^3 + 48x^4) + z^{-6}(17x^3 + 48x^4) \\
& + 18z^7x^{\frac{7}{2}} + 18z^{-7}x^{\frac{7}{2}} + 31z^8x^4 + 31z^{-8}x^4,
\end{aligned} \tag{5.2}$$

up to  $\mathcal{O}(x^5)$ . After taking the additional  $\mathbb{Z}_3$  quotient, several terms are projected out. The remaining terms have only specific powers of  $z$  which is multiples of 3. Thus, we obtain the index of the  $\mathbb{Z}_3$  quotient of the  $(SU(2)_3 \times SU(2)_{-3})/\mathbb{Z}_2$  BLG theory as

$$\begin{aligned}
I_{\text{BLG}, k=3}^{\mathbb{Z}_3 \text{ quotient}}(x, z) &= 1 + 4x + 12x^2 + 24x^3 + 44x^4 \\
& + z^3(4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}}) + z^{-3}(4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}}) \\
& + z^6(17x^3 + 48x^4) + z^{-6}(17x^3 + 48x^4).
\end{aligned} \tag{5.3}$$

Comparing this with the ABJM indices (5.1), the BLG index can be written as

$$\begin{aligned}
I_{\text{BLG}, k=3}^{\mathbb{Z}_3 \text{ quotient}}(x, z) &= I_{\text{ABJM}, k=3}^{(T=0)}(x) + I_{\text{ABJM}, k=3}^{(T=1)}(x)z^3 + I_{\text{ABJM}, k=3}^{(T=-1)}(x)z^{-3} \\
& + I_{\text{ABJM}, k=3}^{(T=2)}(x)z^6 + I_{\text{ABJM}, k=3}^{(T=-2)}(x)z^{-6}.
\end{aligned} \tag{5.4}$$

Thus, we find that the proposal (1.1) of [10] is correct for  $k = 3$  at least up to  $\mathcal{O}(x^5)$ .

We also show the results for other values of  $k$  in Appendix B.1 and B.2. From Tables 4, 6, 7 and 9, we can easily find

$$I_{\text{BLG}, k=5}^{\mathbb{Z}_5 \text{ quotient}}(x, z) = 1 + 4x + 12x^2 + 32x^3 + (6x^{\frac{5}{2}} + 28x^{\frac{7}{2}})z^5 + (6x^{\frac{5}{2}} + 28x^{\frac{7}{2}})z^{-5}$$

$$\begin{aligned}
&= I_{\text{ABJM},k=5}^{(T=0)}(x) + I_{\text{ABJM},k=5}^{(T=1)}(x)z^5 + I_{\text{ABJM},k=5}^{(T=-1)}(x)z^{-5} \\
I_{\text{BLG},k=7}^{\mathbb{Z}_7 \text{ quotient}}(x, z) &= 1 + 4x + 12x^2 + 8x^3 + 12x^4 + 8x^{\frac{7}{2}}z^7 + 8x^{\frac{7}{2}}z^{-7} \\
&= I_{\text{ABJM},k=7}^{(T=0)}(x) + I_{\text{ABJM},k=7}^{(T=1)}(x)z^7 + I_{\text{ABJM},k=7}^{(T=-1)}(x)z^{-7}, \tag{5.5}
\end{aligned}$$

up to  $\mathcal{O}(x^5)$ . Again we can see again that the precise matching is revealed after we impose the additional identification  $\mathbb{Z}_k$  and topological charge  $T$  of the ABJM theory has the one-to-one correspondence with the  $U(1)_t$  charge of the BLG theory. By contrast, there are no matching for  $k = 6$  from Tables 5 and 8. These are consistent with the conjecture (1.1).

## 5.2 $N = 3$

As we have seen in Section 2, the  $U(3)_k \times U(3)_{-k}$  ABJM theory<sup>8</sup> would also be isomorphic to the  $\mathbb{Z}_k$  quotient of the  $(SU(3)_k \times SU(3)_{-k})/\mathbb{Z}_3$  theory. Since the expected isomorphism (1.3) can hold iff  $N$  and  $k$  are coprime,  $k$  must not be multiples of 3 in this case.

First let us consider the case for  $k = 1$ . From Table 18 in Appendix B.4, we find the index of the  $(SU(3)_1 \times SU(3)_{-1})/\mathbb{Z}_3$  theory as

$$\begin{aligned}
I_{SU(3),k=1}(x, z) &= 1 + 8x + 71x^2 + 320x^3 + (2x^{1/2} + 24x^{3/2} + 156x^{5/2})z \\
&\quad + (6x + 56x^2 + 293x^3)z^2 + (14x^{3/2} + 114x^{5/2})z^3 \tag{5.6}
\end{aligned}$$

up to  $\mathcal{O}(x^4)$ . Since the additional  $\mathbb{Z}_{k=1}$  identification for this case is trivial, we can easily see from Table 13 in Appendix B.3 that

$$\begin{aligned}
I_{SU(3),k=1}^{\mathbb{Z}_1 \text{ quotient}}(x, z) &= I_{\text{ABJM},k=1}^{(T=0)}(x) + I_{\text{ABJM},k=1}^{(T=1)}(x)z^1 + I_{\text{ABJM},k=1}^{(T=-1)}(x)z^{-1} + I_{\text{ABJM},k=1}^{(T=2)}(x)z^2 \\
&\quad + I_{\text{ABJM},k=1}^{(T=-2)}(x)z^{-2} + I_{\text{ABJM},k=1}^{(T=3)}(x)z^3 + I_{\text{ABJM},k=1}^{(T=-3)}(x)z^{-3}. \tag{5.7}
\end{aligned}$$

The results for other values are also presented in Appendix B.4 and B.3. From Tables 14, 16, 17, 19, 21 and 22, we also find

$$\begin{aligned}
I_{SU(3),k=2}^{\mathbb{Z}_2 \text{ quotient}}(x, z) &= 1 + 4x + 21x^2 + 92x^3 + (3x + 16x^2 + 87x^3)z^2 + (3x + 16x^2 + 87x^3)z^{-2} \\
&\quad + (11x^2 + 60x^3)z^4 + (11x^2 + 60x^3)z^{-4} \\
&= I_{\text{ABJM},k=2}^{(T=0)}(x) + I_{\text{ABJM},k=2}^{(T=1)}(x)z^2 + I_{\text{ABJM},k=2}^{(T=-1)}(x)z^{-2} \\
&\quad + I_{\text{ABJM},k=2}^{(T=2)}(x)z^4 + I_{\text{ABJM},k=2}^{(T=-4)}(x)z^{-4} \\
I_{SU(3),k=4}^{\mathbb{Z}_4 \text{ quotient}}(x, z) &= 1 + 4x + 12x^2 + 32x^3 + (5x^2 + 24x^3)z^4 + (5x^2 + 24x^3)z^{-4} \\
&= I_{\text{ABJM},k=4}^{(T=0)}(x) + I_{\text{ABJM},k=4}^{(T=1)}(x)z^4 + I_{\text{ABJM},k=4}^{(T=-1)}(x)z^{-4} \\
I_{SU(3),k=5}^{\mathbb{Z}_5 \text{ quotient}}(x, z) &= 1 + 4x + 12x^2 + 32x^3 + 6x^{5/2}z^5 + 6x^{5/2}z^{-5} \\
&= I_{\text{ABJM},k=5}^{(T=0)}(x) + I_{\text{ABJM},k=5}^{(T=1)}(x)z^5 + I_{\text{ABJM},k=5}^{(T=-1)}(x)z^{-5} \tag{5.8}
\end{aligned}$$

up to  $\mathcal{O}(x^4)$ . Therefore, we find that the isomorphism (1.3) for  $N = 3$  is also correct for various values of  $k$  at least up to  $\mathcal{O}(x^4)$  while there are no matching for  $k = 3$  from Tables 15 and 20. This is consistent with the conjecture (1.3).

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<sup>8</sup>The indices of the  $U(3)_k \times U(3)_{-k}$  ABJM theory for  $k = 1, 2$  have been also calculated in [24]. See also [25].

## 6. Search for a possibility of extended isomorphism with higher $k$

In this section, let us consider whether an extension of the isomorphisms (1.2) and (1.4) to higher  $k$  is possible or not. We compute the superconformal indices of theories with an identical moduli space and compare the results of these theories.

### 6.1 $U(2+l)_k \times U(2)_{-k}$ ABJ theory v.s. $(SU(2)_{k^2} \times SU(2)_{-k^2})/\mathbb{Z}_2$ BLG theory

If two theories are isomorphic, these theories should have a same moduli space. Since the moduli space of the  $(SU(2)_{k^2} \times SU(2)_{-k^2})/\mathbb{Z}_2$  BLG theory is same as the one of the  $U(2+l)_k \times U(2)_{-k}$  ABJ theory ( $0 \leq l \leq |k|$ ), these pairs would be candidates for the extension of the isomorphism (1.4) with higher  $k$ . Actually if we take  $k = 2$  and  $l = 1$ , this is nothing but the pair of (1.4).

First, let us consider the case for  $k = 3$ . Then, the  $(SU(2)_9 \times SU(2)_{-9})/\mathbb{Z}_2$  BLG theory has the same moduli space with the  $U(2)_3 \times U(2)_{-3}$ ,  $U(3)_3 \times U(2)_{-3}$ ,  $U(4)_3 \times U(2)_{-3}$  and  $U(5)_3 \times U(2)_{-3}$  ABJ theories. Since it is widely believed that  $U(4)_3 \times U(2)_{-3}$  and  $U(5)_3 \times U(2)_{-3}$  ABJ theories are equivalent to the  $U(2)_3 \times U(2)_{-3}$  and  $U(3)_3 \times U(2)_{-3}$  theories via parity duality [2], we can concentrate only on the  $U(2)_3 \times U(2)_{-3}$  and  $U(3)_3 \times U(2)_{-3}$  theories. From Table 11, we can pick up the result of the  $(SU(2)_9 \times SU(2)_{-9})/\mathbb{Z}_2$  BLG theory as

$$\begin{aligned} I_{\text{BLG}, k=9}(x, z) = & 1 + 4x + 12x^2 + 8x^3 + 12x^4 + (3x + 8x^2 + 12x^3 + 8x^4)z^2 \\ & + (3x + 8x^2 + 12x^3 + 8x^4)z^{-2} + (6x^2 + 12x^3 + 12x^4)z^4 \\ & + (6x^2 + 12x^3 + 12x^4)z^{-4}, \end{aligned} \quad (6.1)$$

up to  $\mathcal{O}(x^5)$ . On the other hand, the results of the  $U(2)_3 \times U(2)_{-3}$  and  $U(3)_3 \times U(2)_{-3}$  ABJ(M) theories from Tables 1 and 24 are

$$\begin{aligned} I_{U(2) \times U(2), k=3}^{(T=0)}(x) &= 1 + 4x + 12x^2 + 24x^3 + 44x^4, \\ I_{U(2) \times U(2), k=3}^{(T=1)}(x) &= 4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 22x^{\frac{7}{2}}, \\ I_{U(2) \times U(2), k=3}^{(T=2)}(x) &= 17x^3 + 48x^4, \end{aligned} \quad (6.2)$$

$$\begin{aligned} I_{U(3) \times U(2), k=3}^{(T=0)}(x) &= 1 + 4x + 12x^2 + 28x^3 + 37x^4, \\ I_{U(3) \times U(2), k=3}^{(T=1)}(x) &= 4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 26x^{\frac{7}{2}}, \\ I_{U(3) \times U(2), k=3}^{(T=2)}(x) &= 17x^3 + 48x^4. \end{aligned} \quad (6.3)$$

We can easily see that the result of the BLG theory does not match with the calculations of the  $U(2)_3 \times U(2)_{-3}$  and the  $U(3)_3 \times U(2)_{-3}$  ABJ theories. In particular, there are several terms whose powers of  $z$  does not corresponds to integer  $T$  in the ABJ theories. Therefore, although the  $(SU(2)_9 \times SU(2)_{-9})/\mathbb{Z}_2$  BLG theory has the same moduli space with the  $U(2+l)_3 \times U(2)_{-3}$  ABJ theory, there are no isomorphisms among these theories.

Next, let us consider a case for  $k = 4$ . Then, the  $(SU(2)_{16} \times SU(2)_{-16})/\mathbb{Z}_2$  BLG theory has the same moduli space with the  $U(2+l)_4 \times U(2)_{-4}$  ( $l = 0, 1, \dots, 4$ ) ABJ

theories. Similar to the  $k = 3$  case, we can concentrate on the indices for  $l = 0, 1, 2$ . Table 12 shows the result of the  $(SU(2)_{16} \times SU(2)_{-16})/\mathbb{Z}_2$  BLG theory. Again we can easily see that the results do not agree with each other.

$$\begin{aligned} I_{\text{BLG}, k=16}(x, z) = & 1 + 4x + 12x^2 + 8x^3 + 12x^4 + 15x^4z^8 + 15x^4z^{-8} + (16x^4 + 10x^3)z^6 \\ & + (16x^4 + 10x^3)z^{-6} + (12x^4 + 12x^3 + 6x^2)z^4 + (12x^4 + 12x^3 + 6x^2)z^{-4} \\ & + (8x^4 + 12x^3 + 8x^2 + 3x)z^2 + (8x^4 + 12x^3 + 8x^2 + 3x)z^{-2}, \end{aligned} \quad (6.4)$$

up to  $\mathcal{O}(x^5)$ . From Tables 3, 25 and 26, we pick up the results of the ABJ theories as

$$\begin{aligned} I_{U(2) \times U(2), k=4}^{(T=0)}(x) &= 1 + 4x + 12x^2 + 8x^3 + 37x^4, \\ I_{U(2) \times U(2), k=4}^{(T=1)}(x) &= 5x^2 + 24x^3 + 23x^4, \\ I_{U(2) \times U(2), k=4}^{(T=2)}(x) &= 17x^3 + 48x^4, \end{aligned} \quad (6.5)$$

$$\begin{aligned} I_{U(3) \times U(2), k=4}^{(T=0)}(x) &= 1 + 4x + 12x^2 + 12x^3 + 30x^4, \\ I_{U(3) \times U(2), k=4}^{(T=1)}(x) &= 5x^2 + 24x^3 + 28x^4, \\ I_{U(3) \times U(2), k=4}^{(T=2)}(x) &= 24x^4, \end{aligned} \quad (6.6)$$

$$\begin{aligned} I_{U(4) \times U(2), k=4}^{(T=0)}(x) &= 1 + 4x + 12x^2 + 12x^3 + 31x^4, \\ I_{U(4) \times U(2), k=4}^{(T=1)}(x) &= 5x^2 + 24x^3 + 28x^4. \end{aligned} \quad (6.7)$$

Thus, we conclude that extension of the isomorphism (1.4) to higher  $k$  seems to be impossible.

## 6.2 $U(2+l)_k \times U(2)_{-k}$ ( $0 \leq l \leq |k|$ ) ABJ theory v.s. $SU(2)_{k^2/2} \times SU(2)_{-k^2/2}$ BLG theory

Similarly we can find the pairs with same moduli spaces: the  $SU(2)_{k^2/2} \times SU(2)_{-k^2/2}$  BLG theory and the  $U(2+l)_k \times U(2)_{-k}$  ( $0 \leq l \leq |k|$ ) ABJ theories. Actually if we set  $k = 4$  and  $l = 0$ , this becomes the pair of (1.2). First, let us consider the case for  $k = 4$ . Then the  $SU(2)_8 \times SU(2)_{-8}$  BLG theory has a same moduli space with the  $U(2+l)_4 \times U(2)_{-4}$  ( $l = 0, 1, 2, 3, 4$ ) ABJ theories. In this case, we can restrict to the cases for  $l = 0, 1, 2$  as before. Table 10 shows the result of the  $SU(2)_8 \times SU(2)_{-8}$  BLG theory, which is given by

$$\begin{aligned} I_{\text{BLG}, k=8}(x, z) = & 1 + 4x + 12x^2 + 8x^3 + 12x^4 \\ & + (3x + 8x^2 + 12x^3 + 8x^4)z^2 + (3x + 8x^2 + 12x^3 + 8x^4)z^{-2} \\ & + (6x^2 + 12x^3 + 12x^4)z^4 + (6x^2 + 12x^3 + 12x^4)z^{-4}. \end{aligned} \quad (6.8)$$

Comparing this with (6.5), (25) and (26), we can easily see again that the results do not match with each other. Thus, we conclude that extension of the isomorphism (1.2) to higher  $k$  might be impossible.



## 7. Conclusions

In this paper we calculated the superconformal indices of the  $U(N_1)_k \times U(N_2)_{-k}$  ABJ(M) theories and  $(SU(N)_k \times SU(N)_{-k})/\mathbb{Z}_N$  theories including the BLG theories for various values of the rank and the Chern-Simons level. We utilize the indices to test the conjectured isomorphism between several M2-brane theories beyond the classical moduli space analysis. Actually we have been confirmed the isomorphism between

$$U(2)_k \times U(2)_{-k} \text{ ABJM and } \mathbb{Z}_k \text{ quotient of } (SU(2)_k \times SU(2)_{-k})/\mathbb{Z}_2 \text{ BLG theory}$$

for the cases without the  $\mathcal{N} = 8$  SUSY enhancement. Since the  $(SU(2) \times SU(2))/\mathbb{Z}_2$  theory can be expressed by the  $A_4$  BLG theory, this verification enables us to understand the significance of the  $A_4$  BLG theory with the higher Chern-Simons level  $k > 2$ . By comparing the indices with the fixed topological charge of the ABJM theory with the contributions from the corresponding charge of the BLG theory, we have been obtained the clear understanding for the correspondence. We have also tested the conjectured equivalence between

$$U(3)_k \times U(3)_{-k} \text{ ABJM and } \mathbb{Z}_k \text{ quotient of } (SU(3)_k \times SU(3)_{-k})/\mathbb{Z}_3 \text{ BLG theory}$$

and it turns out that the isomorphism holds for various values of  $k$  at least up to  $\mathcal{O}(x^4)$ . Moreover we investigated a possibility of extensions of isomorphisms (i)  $U(2)_2 \times U(2)_{-2}$  ABJM and  $SU(2)_2 \times SU(2)_{-2}$  BLG theory and (ii)  $U(3)_2 \times U(2)_{-2}$  ABJ theory and  $(SU(2)_4 \times SU(2)_{-4})/\mathbb{Z}_2$  BLG theory to higher  $k$ . Comparing the indices of theories with an identical moduli space, we have found that such extensions might be impossible.

Important subject related to our work is concerned with the  $-3\lambda^2/8N^2$  discrepancy in the  $AdS_4/CFT_3$  correspondence. In [26], apart from the worldsheet instanton contributions and the constant map contributions, the all genus free energy of the ABJM matrix model was resummed to the Airy function which depends on the “renormalized” t’Hooft coupling  $\lambda_{\text{ren}}$  given by

$$\lambda_{\text{ren}} = \lambda - \frac{1}{24} - \frac{\lambda^2}{3N^2}. \quad (7.1)$$

This shift was originally observed in [27] by simplifying the expression of the all genus free energy. Note that this renormalization is consistent with the Fermi gas approach [28], numerical calculation [29] and exact calculation for  $k = 1$  [30, 31]. However, this renormalization of the t’Hooft coupling is slightly different from the expectation from the gravity side [32]:

$$\lambda_{\text{ren}}^{\text{grav}} = \lambda - \frac{1}{24} + \frac{\lambda^2}{24N^2}. \quad (7.2)$$

This shift comes from the higher curvature correction  $C_3 \wedge I_8$  in M-theory. Here  $I_8$  is a 8-form anomaly polynomial [33]. Although (7.1) and (7.2) agree in the large  $N$  limit, there is a discrepancy  $-3\lambda^2/8N^2$  at the non-planar level. From the aspect of testing  $AdS_4/CFT_3$

duality in quantum level, we should definitely obtain more understanding on both the gauge theory side and the gravity side. As discussed in [26], a possible resolution on the matrix model side is to consider the effect of  $U(1)$  factors in the gauge group  $U(N) \times U(N)$  which provide finite  $N$  correction. Although the current status of this problem is unclear, it is worth revisiting the  $U(1)$  factors in ABJM theory in greater detail.

Recently there have been some arguments about applying the Lie 3-algebra to the M5-branes [34] (see also [35, 36]). Although the significance of the Lie 3-algebra in 6d  $\mathcal{N} = (2, 0)$  theory is not so clear, it is valuable to keep in mind the role of the  $U(1)$  factors.

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### A. partition function of $SU(2)_k \times SU(2)_{-k}$ ABJM theory

Here we provide a further evidence for the conjecture (1.2) by calculating the partition function of  $SU(2)_k \times SU(2)_{-k}$  ABJM theory on  $S^3$ . The partition function of  $U(2)_k \times U(2)_{-k}$  ABJM theory has been exactly calculated in [37].

In [38], the localization technique was applied to the ABJM theory on  $S^3$  and its partition function was shown to be reduced to a matrix integral

$$Z(N, k) = \frac{1}{(N!)^2} \int \frac{d^N \mu}{(2\pi)^N} \frac{d^N \nu}{(2\pi)^N} \times \frac{\prod_{i < j} \left( 2 \sinh \frac{\mu_i - \mu_j}{2} \right)^2 \left( 2 \sinh \frac{\nu_i - \nu_j}{2} \right)^2}{\prod_{i, j} \left( 2 \cosh \frac{\mu_i - \nu_j}{2} \right)^2} \exp \left[ \frac{ik}{4\pi} \sum_{i=1}^N (\mu_i^2 - \nu_i^2) \right], \quad (\text{A.1})$$

which is commonly referred to as the ABJM matrix model. Here we consider the constraints

$$\mu_1 + \mu_2 = 0, \quad \nu_1 + \nu_2 = 0, \quad (\text{A.2})$$

for picking up the  $SU(2)_k \times SU(2)_{-k}$  factor from the  $U(2)_k \times U(2)_{-k}$  ABJM matrix model. Then the partition function of the  $SU(2)_k \times SU(2)_{-k}$  theory is given by

$$\begin{aligned} & Z_{SU(2)_k \times SU(2)_{-k}} \\ &= \frac{1}{2!} \sum_{\sigma} (-1)^{\sigma} \int \frac{d^2 \mu}{(2\pi)^2} \frac{d^2 \nu}{(2\pi)^2} \prod_i \frac{\exp \left[ \frac{ik}{4\pi} \sum_{i=1}^2 (\mu_i^2 - \nu_i^2) \right] [2\pi\delta(\mu_1 + \mu_2)] [2\pi\delta(\nu_1 + \nu_2)]}{\left[ 2 \cosh \left( \frac{\mu_i - \nu_i}{2} \right) \right] \left[ 2 \cosh \left( \frac{\mu_i - \nu_{\sigma(i)}}{2} \right) \right]} \\ &= \frac{1}{32} \int \frac{d\mu}{2\pi} \frac{d\nu}{2\pi} \left[ \frac{1}{\cosh^4 \left( \frac{\mu - \nu}{2} \right)} - \frac{1}{\cosh^2 \left( \frac{\mu - \nu}{2} \right) \cosh^2 \left( \frac{\mu + \nu}{2} \right)} \right] \exp \left[ \frac{ik}{2\pi} (\mu^2 - \nu^2) \right]. \quad (\text{A.3}) \end{aligned}$$

After taking a change of variables

$$\lambda = \frac{\mu - \nu}{2}, \quad \lambda' = \frac{\mu + \nu}{2}, \quad (\text{A.4})$$

we finally obtain

$$\begin{aligned}
Z_{SU(2)_k \times SU(2)_{-k}} &= \frac{1}{32} \int 2 \frac{d\lambda}{2\pi} \frac{d\lambda'}{2\pi} \left[ \frac{1}{\cosh^4 \lambda} - \frac{1}{\cosh^2 \lambda \cosh^2 \lambda'} \right] \exp \left[ \frac{2ik}{\pi} \lambda \lambda' \right] \\
&= \frac{1}{64k} - \frac{k}{64} \int d\lambda \frac{2\lambda}{\cosh^2 \pi \lambda \sinh \pi k \lambda} \\
&= \frac{k}{32} \int_{-\infty}^{\infty} d\lambda \frac{\lambda}{\sinh \pi k \lambda} \tanh^2 \pi \lambda \\
&= \frac{k}{2} Z_{U(2)_k \times U(2)_{-k}}.
\end{aligned} \tag{A.5}$$

Particularly for  $k = 2$ , we find

$$Z_{U(2)_2 \times U(2)_{-2}} = Z_{SU(2)_2 \times SU(2)_{-2}}. \tag{A.6}$$

This is consistent with the expected isomorphism between the  $U(2)_2 \times U(2)_{-2}$  ABJM theory and  $SU(2)_2 \times SU(2)_{-2}$  BLG theory.

## B. Full result

Here we show our results for the superconformal indices of various M2-brane theories.

### B.1 $U(2) \times U(2)$ ABJM theory

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 8x^3 + 37x^4$
$ 0, 0\rangle 0, 0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$
$ 1, -1\rangle 1, -1\rangle$	$25x^4$
$T = 1$	$5x^2 + 24x^3 + 23x^4$
$ 1, 0\rangle 1, 0\rangle$	$5x^2 + 24x^3 + 23x^4$
$T = 2$	$17x^3 + 48x^4$
$ 1, 1\rangle 1, 1\rangle$	$10x^3 + 16x^4$
$ 2, 0\rangle 2, 0\rangle$	$7x^3 + 32x^4$
total	$1 + 4x + 8x^{\frac{3}{2}} + 12x^2 + 40x^{\frac{5}{2}} + 58x^3 + 44x^{\frac{7}{2}} + 140x^4$

Table 3:  $U(2)_4 \times U(2)_{-4}$ .

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$
$ 0, 0\rangle 0, 0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$
$T = 1$	$6x^{\frac{5}{2}} + 28x^{\frac{7}{2}}$
$ 1, 0\rangle 1, 0\rangle$	$6x^{\frac{5}{2}} + 28x^{\frac{7}{2}}$

Table 4:  $U(2)_5 \times U(2)_{-5}$ .

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$
$ 0,0\rangle 0,0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$
$T = 1$	$7x^2 + 32x^4$
$ 1,0\rangle 1,0\rangle$	$7x^2 + 32x^4$

Table 5:  $U(2)_6 \times U(2)_{-6}$ .

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$
$ 0,0\rangle 0,0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$
$T = 1$	$8x^{\frac{7}{2}}$
$ 1,0\rangle 1,0\rangle$	$8x^{\frac{7}{2}}$

Table 6:  $U(2)_7 \times U(2)_{-7}$ .

## B.2 $(SU(2) \times SU(2))/\mathbb{Z}_2$ BLG theory

GNO charges	Index contribution
$ 0,0\rangle 0,0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4 +$ $z^2(3x + 8x^2 + 12x^3 + 8x^4) + z^{-2}(3x + 8x^2 + 12x^3 + 8x^4) +$ $z^4(6x^2 + 12x^3 + 12x^4) + z^{-4}(6x^2 + 12x^3 + 12x^4) +$ $z^6(10x^3 + 16x^4) + z^{-6}(10x^3 + 16x^4) + 15z^4x^4 + 15z^{-8}x^4$
$ 1/2, -1/2\rangle 1/2, -1/2\rangle$	$z(12x^{\frac{5}{2}} + 34x^{\frac{7}{2}}) + z^{-1}(12x^{\frac{5}{2}} + 34x^{\frac{7}{2}}) +$ $z^3(10x^{\frac{5}{2}} + 32x^{\frac{7}{2}}) + z^{-3}(10x^{\frac{5}{2}} + 32x^{\frac{7}{2}}) +$ $z^5(6x^{\frac{5}{2}} + 28x^{\frac{7}{2}}) + z^{-5}(6x^{\frac{5}{2}} + 28x^{\frac{7}{2}}) + 14z^7x^{\frac{7}{2}} + 14z^{-7}x^{\frac{7}{2}}$

Table 7:  $(SU(2)_5 \times SU(2)_{-5})/\mathbb{Z}_2$

GNO charges	Index contribution
$ 0\rangle 0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4 +$ $z^2(3x + 8x^2 + 12x^3 + 8x^4) + z^{-2}(3x + 8x^2 + 12x^3 + 8x^4) +$ $z^4(6x^2 + 12x^3 + 12x^4) + z^{-4}(6x^2 + 12x^3 + 12x^4) +$ $z^6(10x^3 + 16x^4) + z^{-6}(10x^3 + 16x^4) + 15z^4x^4 + 15z^{-8}x^4$
$ 1/2\rangle 1/2\rangle$	$16x^3 + 41x^4 + z^2(15x^3 + 40x^4) + z^{-2}(15x^3 + 40x^4)$ $z^4(12x^3 + 37x^4) + z^{-4}(12x^3 + 37x^4) +$ $z^6(7x^3 + 32x^4) + z^{-6}(7x^3 + 32x^4) + 16z^8x^4 + 16z^{-8}x^4$

Table 8:  $(SU(2)_6 \times SU(2)_{-6})/\mathbb{Z}_2$

GNO charges	Index contribution
$ 0\rangle 0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4 +$ $z^2(3x + 8x^2 + 12x^3 + 8x^4) + z^{-2}(3x + 8x^2 + 12x^3 + 8x^4) +$ $z^4(6x^2 + 12x^3 + 12x^4) + z^{-4}(6x^2 + 12x^3 + 12x^4) +$ $z^6(10x^3 + 16x^4) + z^{-6}(10x^3 + 16x^4) + 15z^4x^4 + 15z^{-8}x^4$
$ 1/2\rangle 1/2\rangle$	$20zx^{\frac{7}{2}} + 20z^{-1}x^{\frac{7}{2}} + 18z^3x^{\frac{7}{2}} + 18z^{-3}x^{\frac{7}{2}} +$ $14z^5x^{\frac{7}{2}} + 14z^{-5}x^{\frac{7}{2}} + 8z^7x^{\frac{7}{2}} + 8z^{-7}x^{\frac{7}{2}}$

Table 9:  $(SU(2)_7 \times SU(2)_{-7})/\mathbb{Z}_2$

GNO charges	Index contribution
$ 0\rangle 0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$ $+ (3x + 8x^2 + 12x^3 + 8x^4)z^2 + (3x + 8x^2 + 12x^3 + 8x^4)z^{-2}$ $+ (6x^2 + 12x^3 + 12x^4)z^4 + (6x^2 + 12x^3 + 12x^4)z^{-4}$

Table 10:  $(SU(2)_8 \times SU(2)_{-8})/\mathbb{Z}_2$

GNO charges	Index contribution
$ 0\rangle 0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4$ $+ (3x + 8x^2 + 12x^3 + 8x^4)z^2 + (3x + 8x^2 + 12x^3 + 8x^4)z^{-2}$ $+ (6x^2 + 12x^3 + 12x^4)z^4 + (6x^2 + 12x^3 + 12x^4)z^{-4}$

Table 11:  $(SU(2)_9 \times SU(2)_{-9})/\mathbb{Z}_2$

GNO charges	Index contribution
$ 0\rangle 0\rangle$	$1 + 4x + 12x^2 + 8x^3 + 12x^4 + 15x^4z^8 + 15x^4z^{-8}$ $+ (16x^4 + 10x^3)z^6 + (16x^4 + 10x^3)z^{-6}$ $+ (12x^4 + 12x^3 + 6x^2)z^4 + (12x^4 + 12x^3 + 6x^2)z^{-4}$ $+ (8x^4 + 12x^3 + 8x^2 + 3x)z^2 + (8x^4 + 12x^3 + 8x^2 + 3x)z^{-2}$

Table 12:  $(SU(2)_{16} \times SU(2)_{-16})/\mathbb{Z}_2$

### B.3 $U(3) \times U(3)$ ABJM theory

GNO charges	Index contribution
$T = 0$	$1 + 8x + 71x^2 + 320x^3$
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$
$ 1, 0, -1\rangle 1, 0, -1\rangle$	$4x + 32x^2 + 92x^3$
$ 2, 0, -2\rangle 2, 0, -2\rangle$	$9x^2 + 60x^3$
$ 1, 1, -2\rangle 1, 1, -2\rangle$	$9x^2 + 36x^3$

$ -1, -1, 2\rangle -1, -1, 2\rangle$	$9x^2 + 36x^3$
$ 1, 2, -3\rangle 1, 2, -3\rangle$	$24x^3$
$ -1, -2, 3\rangle -1, -2, 3\rangle$	$24x^3$
$ 3, 0, -3\rangle 3, 0, -3\rangle$	$16x^3$
$T = 1$	$2x^{1/2} + 24x^{3/2} + 156x^{5/2}$
$ 1, 0, 0\rangle 1, 0, 0\rangle$	$2x^{1/2} + 12x^{3/2} + 42x^{5/2}$
$ 1, 1, -1\rangle 1, 1, -1\rangle$	$6x^{3/2} + 28x^{5/2}$
$ 2, 0, -1\rangle 2, 0, -1\rangle$	$6x^{3/2} + 44x^{5/2}$
$ 2, 1, -2\rangle 2, 1, -2\rangle$	$18x^{5/2}$
$ 3, 0, -2\rangle 3, 0, -2\rangle$	$12x^{5/2}$
$ 3, -1, -1\rangle 3, -1, -1\rangle$	$12x^{5/2}$
$T = 2$	$6x + 56x^2 + 311x^3$
$ 2, 0, 0\rangle 2, 0, 0\rangle$	$3x + 16x^2 + 52x^3$
$ 1, 1, 0\rangle 1, 1, 0\rangle$	$3x + 20x^2 + 51x^3$
$ 2, 1, -1\rangle 2, 1, -1\rangle$	$12x^2 + 64x^3$
$ 3, 0, -1\rangle 3, 0, -1\rangle$	$8x^2 + 56x^3$
$ 3, 1, -2\rangle 3, 1, -2\rangle$	$24x^3$
$ 2, 2, -2\rangle 2, 2, -2\rangle$	$18x^3$
$ 4, 0, -2\rangle 4, 0, -2\rangle$	$15x^3$
$ 4, -1, -1\rangle 4, -1, -1\rangle$	$15x^3$
$T = 3$	$14x^{3/2} + 114x^{5/2}$
$ 3, 0, 0\rangle 3, 0, 0\rangle$	$4x^{3/2} + 20x^{5/2}$
$ 2, 1, 0\rangle 2, 1, 0\rangle$	$6x^{3/2} + 44x^{5/2}$
$ 1, 1, 1\rangle 1, 1, 1\rangle$	$4x^{3/2} + 12x^{5/2}$
$ 3, 1, -1\rangle 3, 1, -1\rangle$	$16x^{5/2}$
$ 2, 2, -1\rangle 2, 2, -1\rangle$	$12x^{5/2}$
$ 4, 0, -1\rangle 4, 0, -1\rangle$	$10x^{5/2}$

Table 13:  $U(3)_1 \times U(3)_{-1}$ .

GNO charges	Index contribution
$T = 0$	$1 + 4x + 21x^2 + 92x^3$
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$
$ 1, 0, -1\rangle 1, 0, -1\rangle$	$9x^2 + 60x^3$
$T = 1$	$3x + 16x^2 + 87x^3$
$ 1, 0, 0\rangle 1, 0, 0\rangle$	$3x + 16x^2 + 54x^3$
$ 2, 0, -1\rangle 2, 0, -1\rangle$	$15x^3$
$ 1, 1, -1\rangle 1, 1, -1\rangle$	$18x^3$
$T = 2$	$11x^2 + 60x^3$
$ 2, 0, 0\rangle 2, 0, 0\rangle$	$5x^2 + 24x^3$

$ 1, 1, 0\rangle 1, 1, 0\rangle$	$6x^2 + 36x^3$
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Table 14:  $U(3)_2 \times U(3)_{-2}$ .

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 48x^3$
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$
$ 1, 0, -1\rangle 1, 0, -1\rangle$	$16x^3$
$T = 1$	$4x^{3/2} + 20x^{5/2}$
$ 1, 0, 0\rangle 1, 0, 0\rangle$	$4x^{3/2} + 20x^{5/2}$
$T = 2$	$17x^3$
$ 2, 0, 0\rangle 2, 0, 0\rangle$	$7x^3$
$ 1, 1, 0\rangle 1, 1, 0\rangle$	$10x^3$

Table 15:  $U(3)_3 \times U(3)_{-3}$ .

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 32x^3$
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$
$T = 1$	$5x^2 + 24x^3$
$ 1, 0, 0\rangle 1, 0, 0\rangle$	$5x^2 + 24x^3$

Table 16:  $U(3)_4 \times U(3)_{-4}$ .

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 32x^3$
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$
$T = 1$	$6x^{5/2}$
$ 1, 0, 0\rangle 1, 0, 0\rangle$	$6x^{5/2}$

Table 17:  $U(3)_5 \times U(3)_{-5}$ .

#### B.4 $(SU(3) \times SU(3))/\mathbb{Z}_3$ theory

GNO charges	Index contribution
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$ $z^3(4x^{3/2} + 12x^{5/2}) + z^{-3}(4x^{3/2} + 12x^{5/2})$ $10z^6x^3 + 10z^{-6}x^3$
$ \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\rangle \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$	$12z^{-5}x^{5/2} + z(42x^{5/2} + 12x^{3/2} + 2\sqrt{x})$ $+ z^4(36x^3 + 9x^2) + z^{-2}(51x^3 + 20x^2 + 3x)$

$ \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\rangle \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\rangle$	$12z^5x^{5/2} + z^{-1}(42x^{5/2} + 12x^{3/2} + 2\sqrt{x})$ $+z^{-4}(36x^3 + 9x^2) + z^2(51x^3 + 20x^2 + 3x)$
$ \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\rangle \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\rangle$	$2z^5x^{5/2} + z^{-1}(28x^{5/2} + 6x^{3/2}) + z^{-4}(36x^3 + 6x^2) + z^2(52x^3 + 16x^2 + 3x)$
$ \frac{2}{3}, -\frac{2}{3}, \frac{4}{3}\rangle \frac{2}{3}, -\frac{2}{3}, \frac{4}{3}\rangle$	$12z^5x^{5/2} + z(28x^{5/2} + 6x^{3/2}) + z^4(36x^3 + 6x^2) + z^{-2}(52x^3 + 16x^2 + 3x)$
$ 1, 0, -1\rangle 1, 0, -1\rangle$	$4x + 32x^2 + 92x^3$ $+z^3(6x^{3/2} + 44x^{5/2}) + z^{-3}(6x^{3/2} + 44x^{5/2}) + 24z^6x^3 + 24z^{-6}x^3$
$ 1, 1, -2\rangle 1, 1, -2\rangle$	$9x^2 + 36x^3$ $+z^3(4x^{3/2} + 20x^{5/2}) + 12z^{-3}x^{5/2} + 10z^{-6}x^3$
$ -1, -1, 2\rangle -1, -1, 2\rangle$	$9x^2 + 36x^3$ $+z^3(12x^{5/2}) + z^{-3}(4x^{3/2} + 20x^{5/2}) + 10z^{-6}x^3$
$ \frac{1}{3}, \frac{4}{3}, -\frac{5}{3}\rangle \frac{1}{3}, \frac{4}{3}, -\frac{5}{3}\rangle$	$12z^{-5}x^{5/2} + z(44x^{5/2} + 6x^{3/2}) + z^4(56x^3 + 8x^2) + z^{-2}(64x^3 + 12x^2)$
$ \frac{1}{3}, -\frac{4}{3}, \frac{5}{3}\rangle \frac{1}{3}, -\frac{4}{3}, \frac{5}{3}\rangle$	$12z^5x^{5/2} + z^{-1}(44x^{5/2} + 6x^{3/2}) + z^{-4}(56x^3 + 8x^2) + z^2(64x^3 + 12x^2)$
$ \frac{2}{3}, \frac{5}{3}, -\frac{7}{3}\rangle \frac{2}{3}, \frac{5}{3}, -\frac{7}{3}\rangle$	$10z^5x^{5/2} + 18z^{-1}x^{5/2} + 24z^{-4}x^3 + z^2(56x^3 + 8x^2)$
$ \frac{2}{3}, -\frac{5}{3}, \frac{7}{3}\rangle \frac{2}{3}, -\frac{5}{3}, \frac{7}{3}\rangle$	$10z^{-5}x^{5/2} + 18zx^{5/2} + 24z^4x^3 + z^{-2}(56x^3 + 8x^2)$
$ \frac{4}{3}, \frac{4}{3}, -\frac{8}{3}\rangle \frac{4}{3}, \frac{4}{3}, -\frac{8}{3}\rangle$	$12zx^{5/2} + 18z^{-2}x^3 + z^4(24x^3 + 5x^2)$
$ \frac{4}{3}, -\frac{4}{3}, \frac{8}{3}\rangle \frac{4}{3}, -\frac{4}{3}, \frac{8}{3}\rangle$	$12z^{-1}x^{5/2} + 18z^2x^3 + z^{-4}(24x^3 + 5x^2)$
$ \frac{5}{3}, \frac{5}{3}, -\frac{10}{3}\rangle \frac{5}{3}, \frac{5}{3}, -\frac{10}{3}\rangle$	$6z^5x^{5/2} + 15z^2x^3$
$ 2, 0, -2\rangle 2, 0, -2\rangle$	$9x^2 + 60x^3$ $+16z^3x^{5/2} + 16z^{-3}16x^{5/2} + 15z^6x^3 + 15z^{-6}x^3$
$ 1, 2, -3\rangle 1, 2, -3\rangle$	$24x^3 + 10z^3x^{5/2} + 12z^6x^3$
$ -1, -2, 3\rangle -1, -2, 3\rangle$	$24x^3 + 10z^{-3}x^{5/2} + 12z^{-6}x^3$
$ \frac{1}{3}, \frac{7}{3}, -\frac{8}{3}\rangle \frac{1}{3}, \frac{7}{3}, -\frac{8}{3}\rangle$	$12zx^{5/2} + 20z^4x^3 + 24z^{-2}x^3$
$ \frac{2}{3}, \frac{8}{3}, -\frac{10}{3}\rangle \frac{2}{3}, \frac{8}{3}, -\frac{10}{3}\rangle$	$15z^2x^3$
$ \frac{4}{3}, \frac{7}{3}, -\frac{11}{3}\rangle \frac{4}{3}, \frac{7}{3}, -\frac{11}{3}\rangle$	$12z^4x^3$
$ 3, 0, -3\rangle 3, 0, -3\rangle$	$16x^3$
$ 4, -2, -2\rangle 4, -2, -2\rangle$	$7z^{-6}x^3$
$ -4, 2, 2\rangle -4, 2, 2\rangle$	$7z^6x^3$
Total	$1 + 8x + 71x^2 + 320x^3 + (2x^{1/2} + 24x^{3/2} + 156x^{5/2})z$ $+(6x + 56x^2 + 293x^3)z^2 + (14x^{3/2} + 114x^{5/2})z^3$

Table 18:  $(SU(3)_1 \times SU(3)_{-1})/\mathbb{Z}_3$

GNO charges	Index contribution
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3 + z^3(4x^{3/2} + 12x^{5/2})$ $z^{-3}(4x^{3/2} + 12x^{5/2}) + 10z^6x^3 + 10z^{-6}x^3$
$ \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\rangle \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$	$12z^5x^{5/2} + z^{-1}(30x^{5/2} + 6x^{3/2})$ $+z^{-4}(36x^3 + 6x^2) + z^2(54x^3 + 16x^2 + 3x)$
$ \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\rangle \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\rangle$	$12z^{-5}x^{5/2} + z(30x^{5/2} + 6x^{3/2})$ $+z^4(36x^3 + 6x^2) + z^{-2}(54x^3 + 16x^2 + 3x)$
$ \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\rangle \frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\rangle$	$12zx^{5/2} + 18z^{-2}x^3 + z^4(24x^3 + 5x^2)$
$ \frac{2}{3}, -\frac{2}{3}, \frac{4}{3}\rangle \frac{2}{3}, -\frac{2}{3}, \frac{4}{3}\rangle$	$12z^{-1}x^{5/2} + 18z^2x^3 + z^{-4}(24x^3 + 5x^2)$



$ 1, 0, -1\rangle 1, 0, -1\rangle$	$16z^3x^{5/2} + 16z^{-3}x^{5/2} + 15z^6x^3 + 15z^{-6}x^3 + 60x^3 + 9x^2$
$ \frac{1}{3}, \frac{4}{3}, -\frac{5}{3}\rangle \frac{1}{3}, \frac{4}{3}, -\frac{5}{3}\rangle$	$15z^2x^3$
$ \frac{1}{3}, -\frac{4}{3}, \frac{5}{3}\rangle \frac{1}{3}, -\frac{4}{3}, \frac{5}{3}\rangle$	$15z^{-2}x^3$
$ 1, 1, -2\rangle 1, 1, -2\rangle$	$7z^6x^3$
$ -1, -1, 2\rangle -1, -1, 2\rangle$	$7z^{-6}x^3$
Total	$1 + 4x + 21x^2 + 92x^3 + (3x + 16x^2 + 87x^3)z^2$ $+ (11x^2 + 60x^3)z^4$

Table 19:  $(SU(3)_2 \times SU(3)_{-2})/\mathbb{Z}_3$

GNO charges	Index contribution
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$ $+ z^3(4x^{3/2} + 12x^{5/2}) + z^{-3}(4x^{3/2} + 12x^{5/2})$ $+ 10z^6x^3 + 10z^{-6}x^3$
$ 1, 0, -1\rangle 1, 0, -1\rangle$	$16x^3$
$ 1/3, 1/3, -2/3\rangle 1/3, 1/3, -2/3\rangle$	$9x^2 + 40x^3 + z^3(4x^{3/2} + 20x^{5/2}) + z^{-3}(12x^{5/2})$ $+ 15z^6x^3 + 10z^{-6}x^3$
$ -1/3, -1/3, 2/3\rangle -1/3, -1/3, 2/3\rangle$	$9x^2 + 40x^3 + z^3(12x^{5/2}) + z^{-3}(4x^{3/2} + 20x^{5/2})$ $+ 10z^6x^3 + 15z^{-6}x^3$
$ 4/3, -2/3, -2/3\rangle 4/3, -2/3, -2/3\rangle$	$7z^{-6}x^3$
$ -4/3, 2/3, 2/3\rangle -4/3, 2/3, 2/3\rangle$	$7z^6x^3$
Total	$1 + 4x + 30x^2 + 128x^3 + z^3(8x^{3/2} + 32x^{5/2}) + 42z^6x^3$

Table 20:  $(SU(3)_3 \times SU(3)_{-3})/\mathbb{Z}_3$

GNO charges	Index contribution
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$ $+ z^3(4x^{3/2} + 12x^{5/2}) + z^{-3}(4x^{3/2} + 12x^{5/2})$ $+ 10z^6x^3 + 10z^{-6}x^3$
$ \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\rangle \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$	$12zx^{5/2} + 18z^{-2}x^3 + z^4(24x^3 + 5x^2)$
$ \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\rangle \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\rangle$	$12z^{-1}x^{5/2} + 18z^2x^3 + z^{-4}(24x^3 + 5x^2)$
Total	$1 + 4x + 12x^2 + 32x^3 + (5x^2 + 24x^3)z^4$

Table 21:  $(SU(3)_4 \times SU(3)_{-4})/\mathbb{Z}_3$

GNO charges	Index contribution
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$ $+ z^3(4x^{3/2} + 12x^{5/2}) + z^{-3}(4x^{3/2} + 12x^{5/2})$ $+ 10z^6x^3 + 10z^{-6}x^3$
$ \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\rangle \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$	$6z^5x^{5/2} + 15z^2x^3$

$ - \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\rangle  - \frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\rangle$	$6z^{-5}x^{5/2} + 15z^{-2}x^3$
Total	$1 + 4x + 12x^2 + 32x^3 + 6x^{5/2}z^5$

Table 22:  $(SU(3)_5 \times SU(3)_{-5})/\mathbb{Z}_3$

GNO charges	Index contribution
$ 0, 0, 0\rangle 0, 0, 0\rangle$	$1 + 4x + 12x^2 + 32x^3$ $+ z^3(4x^{3/2} + 12x^{5/2}) + z^{-3}(4x^{3/2} + 12x^{5/2})$ $+ 10z^6x^3 + 10z^{-6}x^3$
$ 1/3, 1/3, -2/3\rangle 1/3, 1/3, -2/3\rangle$	$7z^6x^3$
$ -1/3, -1/3, 2/3\rangle -1/3, -1/3, 2/3\rangle$	$7z^{-6}x^3$

Table 23:  $(SU(3)_6 \times SU(3)_{-6})/\mathbb{Z}_3$

### B.5 $U(3) \times U(2)$ ABJ theory

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 28x^3 + 37x^4$
$ 0, 0, 0\rangle 0, 0\rangle$	$1 + 4x + 12x^2 + 12x^3 + 5x^4$
$ 1, 0, -1\rangle 1, -1\rangle$	$16x^3 + 32x^4$
$T = 1$	$4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 26x^{\frac{7}{2}}$
$ 1, 0, 0\rangle 1, 0\rangle$	$4x^{\frac{3}{2}} + 20x^{\frac{5}{2}} + 26x^{\frac{7}{2}}$
$T = 2$	$17x^3 + 48x^4$
$ 2, 0, 0\rangle 2, 0\rangle$	$7x^3 + 32x^4$
$ 1, 1, 0\rangle 1, 1\rangle$	$10x^3 + 16x^4$

Table 24:  $U(3)_3 \times U(2)_{-3}$ .

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 12x^3 + 30x^4$
$ 0, 0, 0\rangle 0, 0\rangle$	$1 + 4x + 12x^2 + 12x^3 + 5x^4$
$ 1, 0, -1\rangle 1, -1\rangle$	$25x^4$
$T = 1$	$5x^2 + 24x^3 + 28x^4$
$ 1, 0, 0\rangle 1, 0\rangle$	$5x^2 + 24x^3 + 28x^4$
$T = 2$	$24x^4$
$ 2, 0, 0\rangle 2, 0\rangle$	$9x^4$
$ 1, 1, 0\rangle 1, 1\rangle$	$15x^4$

Table 25:  $U(3)_4 \times U(2)_{-4}$ .

## B.6 $U(4) \times U(2)$ ABJ theory

GNO charges	Index contribution
$T = 0$	$1 + 4x + 12x^2 + 12x^3 + 31x^4$
$ 0, 0, 0, 0\rangle 0, 0\rangle$	$1 + 4x + 12x^2 + 12x^3 + 6x^4$
$ 1, 0, 0, -1\rangle 1, -1\rangle$	$25x^4$
$T = 1$	$5x^2 + 24x^3 + 28x^4$
$ 1, 0, 0, 0\rangle 1, 0\rangle$	$5x^2 + 24x^3 + 28x^4$

Table 26:  $U(4)_4 \times U(2)_{-4}$ .

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